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# Factorization of analytic representations in the unit disc and number-phase statistics of a quantum harmonic oscillator

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**Abstract.** The inner–outer part factorization of analytic representations in the unit disc is used for an effective characterization of the number-phase statistical properties of a quantum harmonic oscillator. It is shown that the factorization is intimately connected with the number-phase Weyl semigroup and its properties. In the Barut–Girardello analytic representation the factorization is implemented as a convolution. Several examples are given which demonstrate the physical significance of the factorization and its role for quantum statistics. In particular, we study the effect of phase-space interference on the factorization properties of a superposition state.

## 1. Introduction

The powerful theory of analytic functions has proved to be very useful in many areas of theoretical physics. Here we use this theory to study the number-phase statistics of a quantum harmonic oscillator. This approach employs the inner–outer part factorization of analytic representations in the unit disc [1, 2]. This representation is based on the Perelomov $SU(1,1)$ coherent states [3]. In the present work we consider the Holstein–Primakoff realization of the $SU(1,1)$ Lie algebra [4, 5], where the $SU(1,1)$ generators act on the Hilbert space of a quantum harmonic oscillator (the Fock space) spanned by the number states $|n\rangle$ ($n = 0, 1, 2, \ldots$). The $SU(1,1)$ coherent states in this case are the eigenstates of the exponential phase operator [6, 7] and are closely related to the problem of the quantum description of oscillator phase [8–15]. (For the present status of the quantum phase problem, see, e.g., [16, 17].) Some other aspects of the $SU(1,1)$ coherent states were studied in [18, 19].

In section 2 we introduce the analytic representations in the unit disc and their 'boundaries' which are the phase-state representations in the unit circle [20, 21]. In section 3 we discuss the factorization of these analytic representations into a product of an inner and an outer function. Important information about a quantum state can be inferred from the factorization of the corresponding analytic function. For example, the phase distribution uniquely determines the outer part, and vice versa. The $SU(1,1)$ coherent states and the Barut–Girardello states [22] are outer states, i.e. their analytic representations are outer functions. All physical properties of such states are uniquely determined by their phase distribution. In section 4 we demonstrate that although the factorization is intimately connected with the analytic representations in the unit disc, it can also be implemented

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in other representations. We consider the Barut–Girardello analytic representation [22] and show that there the factorization becomes convolution. In section 5 we give several examples. It is shown that interference in phase space between components of a quantum superposition state has important effects on the factorization of the corresponding analytic function. We explicitly construct a 'Blaschke state' whose analytic representation in the unit disc is a Blaschke factor. It becomes clear that many aspects of the mathematical theory of Hardy spaces have a physical meaning. In section 6 we demonstrate the connection between the factorization and the phase-space formalism based on the number and phase variables. We consider 'shifted states' and the number-phase Wigner functions. We conclude in section 7 with discussion of our results.

## 2. Analytic representations in the unit disc

It is known that the SU(1, 1) generators $\hat{K}_\pm$ and $\hat{K}_0$ can be realized in the Fock space of a quantum harmonic oscillator. For Bargmann index equal to one half, this realization reads [4, 5]

$$\hat{K}_+ = \hat{N}^{1/2}\hat{a}^\dagger \qquad \hat{K}_- = \hat{a}\hat{N}^{1/2} \qquad \hat{K}_0 = \hat{N} + \tfrac{1}{2} \tag{1}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the usual annihilation and creation operators and $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator. Then the Perelomov SU(1, 1) coherent states are given by

$$|z\rangle = \exp(\xi\hat{K}_+ - \xi^*\hat{K}_-)|0\rangle = (1-|z|^2)^{1/2}\sum_{n=0}^{\infty} z^n|n\rangle \tag{2}$$

$$z = (\xi/|\xi|)\tanh|\xi| \qquad |z| < 1. \tag{3}$$

These states form an overcomplete basis in the Hilbert space [3]. They are the eigenstates of the exponential phase operator

$$\hat{E}_-|z\rangle = z|z\rangle \tag{4}$$

and in this sense they have also been called phase states [12–14]. Some interesting properties of the states $|z\rangle$ were discussed in [18, 19]. The exponential phase operators [6, 7]

$$\hat{E}_- = \sum_{n=0}^{\infty}|n\rangle\langle n+1| \qquad \hat{E}_+ = \hat{E}_-^\dagger = \sum_{n=0}^{\infty}|n+1\rangle\langle n| \tag{5}$$

$$\hat{E}_-|0\rangle = 0 \qquad \hat{E}_-\hat{E}_+ = \hat{1} \qquad \hat{E}_+\hat{E}_- = \hat{1} - |0\rangle\langle 0| \tag{6}$$

are related to $\hat{a}, \hat{a}^\dagger$ through the polar decomposition:

$$\hat{a} = \hat{E}_-\hat{N}^{1/2} \qquad \hat{a}^\dagger = \hat{N}^{1/2}\hat{E}_+. \tag{7}$$

With these definitions, the Holstein–Primakoff realization (1) takes the form

$$\hat{K}_+ = \hat{N}\hat{E}_+ \qquad \hat{K}_- = \hat{E}_-\hat{N} \qquad \hat{K}_0 = \hat{N} + \tfrac{1}{2}. \tag{8}$$

Let $|f\rangle$ be an arbitrary (normalized) state

$$|f\rangle = \sum_{n=0}^{\infty} f_n|n\rangle \qquad \sum_{n=0}^{\infty}|f_n|^2 = 1. \tag{9}$$

Its analytic representation in the unit disc $D(|z| < 1)$ is the function

$$Z(f; z) = (1-|z|^2)^{-1/2}\langle f|z\rangle = \sum_{n=0}^{\infty} f_n^* z^n \tag{10}$$

which belongs to the Hardy space $H_2(D)$ [1, 2]. Various properties and applications of this representation were considered in a number of works [13, 14, 18, 19, 23, 24, 25, 26]. For later use we point out that the number states $|n\rangle$ and the SU(1, 1) coherent states $|z_0\rangle$ are represented respectively by the functions

$$Z(n; z) = z^n \tag{11}$$

$$Z(z_0; z) = \frac{(1 - |z_0|^2)^{1/2}}{1 - z_0^* z}. \tag{12}$$

It is often convenient to use the number and phase representations. One can introduce a 'phase space' based on the number and phase variables and the number-phase Wigner function [27–29]. The number representation of the state $|f\rangle$ given by (9) is the sequence $\{f_n = \langle n|f\rangle\}$. The action of the number operator $\hat{N}$ is represented by multiplication by $n$, while the ladder operator $\hat{E}_+$ shifts the sequence by one place:

$$\hat{N} f_n = n f_n \qquad \hat{E}_+ f_n = f_{n-1}. \tag{13}$$

The phase representation is based on the phase states

$$|\theta\rangle = \lim_{|z|\to 1} (1 - |z|^2)^{-1/2} \left| z = |z| \mathrm{e}^{\mathrm{i}\theta} \right\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}n\theta} |n\rangle. \tag{14}$$

According to equation (4), one has $\hat{E}_- |\theta\rangle = \mathrm{e}^{\mathrm{i}\theta} |\theta\rangle$. The $|\theta\rangle$ states are unnormalizable, non-orthogonal and formally they do not belong to the Hilbert space. However, they do resolve the identity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \; |\theta\rangle\langle\theta| = \hat{1} \tag{15}$$

and the phase-state representation is a useful calculational tool [20, 21]. The normalized state $|f\rangle$ of the form (9) is represented by the periodic function

$$\Theta(f; \theta) = \langle f|\theta\rangle = \lim_{|z|\to 1} Z(f; z = |z| \mathrm{e}^{\mathrm{i}\theta}) = \sum_{n=0}^{\infty} f_n^* \mathrm{e}^{\mathrm{i}n\theta} \tag{16}$$

which is the 'boundary function' of the function $Z(f; z)$. For the sake of simplicity we denote this function as $\Theta(f; \theta)$ instead of $\Theta(f; \mathrm{e}^{\mathrm{i}\theta})$ which would be a more appropriate notation. The limit $|z| \to 1$ in (16) exists for any normalizable state $|f\rangle$. The definition of $\Theta(f; \theta)$ as the boundary function of $Z(f; z)$ is mathematically more rigorous, while the definition as $\langle f|\theta\rangle$ is physically more appealing. By using the identity resolution (15), one finds

$$|f\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \; \Theta^*(f; \theta) |\theta\rangle \tag{17}$$

$$\langle g|f\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \; \Theta(g; \theta) \Theta^*(f; \theta). \tag{18}$$

It is easily seen that the boundary function $\Theta(f; \theta)$ determines uniquely the analytic function $Z(f; z)$:

$$Z(f; z = r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta' \; C(r, \theta - \theta') \Theta(f; \theta') \tag{19}$$

$$C(r, \theta) \equiv (1 - r\mathrm{e}^{\mathrm{i}\theta})^{-1}. \tag{20}$$

The number operator $\hat{N}$ and the ladder operator $\hat{E}_+$ act in the phase representation as

$$\hat{N}\Theta(f;\theta) = -\mathrm{i}\frac{\partial}{\partial\theta}\Theta(f;\theta) \qquad \hat{E}_+\Theta(f;\theta) = \exp(\mathrm{i}\theta)\Theta(f;\theta). \tag{21}$$

The phase properties of a state are determined by its phase distribution [30–33].

$$P(f;\theta) = \frac{1}{2\pi}|\Theta(f;\theta)|^2 = \frac{1}{2\pi}\sum_{n,m=0}^{\infty} f_n^* f_m \mathrm{e}^{\mathrm{i}(n-m)\theta}. \tag{22}$$

This is a positive periodic function of $\theta$ normalized by $\int_{2\pi} \mathrm{d}\theta\; P(f;\theta) = 1$. The number states $|n\rangle$ have the uniform phase distribution $P(n;\theta) = (2\pi)^{-1}$. For the SU(1, 1) coherent states $|z\rangle$ one obtains $P(z;\theta) = (2\pi)^{-1}\mathcal{P}(r,\theta-\phi)$, where $z = r\mathrm{e}^{\mathrm{i}\phi}$ and

$$\mathcal{P}(r,\theta) = \mathrm{Re}\,[2C(r,\theta) - 1] = \frac{1-r^2}{1+r^2-2r\cos\theta} \tag{23}$$

is the Poisson kernel. For later use we also define the harmonic conjugate of $\mathcal{P}(r,\theta)$:

$$\mathcal{Q}(r,\theta) = \mathrm{Im}\,[2C(r,\theta) - 1] = \frac{2r\sin\theta}{1+r^2-2r\cos\theta}. \tag{24}$$

Note the following limits:

$$\lim_{r\to 1}\mathcal{P}(r,\theta) = 2\pi\delta(\theta) \qquad \lim_{r\to 1}\mathcal{Q}(r,\theta) = \cot(\theta/2). \tag{25}$$

The Weyl semigroup for the number and phase operators is introduced [13] as

$$\hat{W}(m,\beta,\gamma) = \hat{E}_+^m \exp(\mathrm{i}\beta\hat{N})\exp(\mathrm{i}\gamma) \tag{26}$$

$$\hat{W}(m_1,\beta_1,\gamma_1)\hat{W}(m_2,\beta_2,\gamma_2) = \hat{W}(m_1+m_2,\beta_1+\beta_2,\gamma_1+\gamma_2+m_2\beta_1) \tag{27}$$

$$\hat{W}(0,0,0) = \hat{1} \qquad \hat{W}^\dagger\hat{W} = \hat{1} \qquad \hat{W}\hat{W}^\dagger = \hat{1} - \sum_{n=0}^{m-1}|n\rangle\langle n| \tag{28}$$

where $\beta$ and $\gamma$ are real parameters and $m$ is a non-negative integer. The operators $\hat{W}$ are isometric but not unitary. They do not have an inverse and therefore they form a semigroup but not a group. (Note that this semigroup becomes effectively a group when the antinormal ordering of the phase operators is applied [28, 32, 34].) The operators $\hat{W}(m,\beta,\gamma)$ play a role similar to that of the Weyl displacement operators $\exp[\mathrm{i}(\mu\hat{x}+\nu\hat{p})]$ in the position-momentum phase space. For example, we mention the following basic properties:

$$\hat{E}_+^m|n\rangle = |n+m\rangle \qquad \exp(\mathrm{i}\beta\hat{N})|n\rangle = \exp(\mathrm{i}\beta n)|n\rangle \tag{29}$$

$$\hat{E}_-^m|z\rangle = z^m|z\rangle \qquad \exp(\mathrm{i}\beta\hat{N})|z\rangle = |z\mathrm{e}^{\mathrm{i}\beta}\rangle. \tag{30}$$

## 3. Factorization of analytic representations in the unit disc into their inner and outer parts

We have explained in the previous section that an arbitrary state can be represented in the Hardy space $H_2(D)$ of analytic functions in the unit disc. An important tool in the theory of these functions is their factorization into the product of an inner and an outer function. An analytic function $Z(f;z)$ can be expressed as [1, 2]

$$Z(f;z) = Z_{\mathrm{in}}(f;z)Z_{\mathrm{out}}(f;z) \tag{31}$$

$$Z_{\mathrm{out}}(f;z) = \exp[\Phi(f;z)] \tag{32}$$

$$\Phi(f; z = re^{i\theta}) = \Phi_R(f; z) + i\Phi_I(f; z)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta' \, [2C(r, \theta - \theta') - 1] \ln|\Theta(f; \theta')| \tag{33}$$

$$\Phi_R(f; z = re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta' \, \mathcal{P}(r, \theta - \theta') \ln|\Theta(f; \theta')| \tag{34}$$

$$\Phi_I(f; z = re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta' \, \mathcal{Q}(r, \theta - \theta') \ln|\Theta(f; \theta')| \tag{35}$$

$$Z_{in}(f; z) = Z(f; z)/Z_{out}(f; z) = Z(f; z) \exp[-\Phi(f; z)]. \tag{36}$$

The functions $Z_{in}(f; z)$ and $Z_{out}(f; z)$ are called the inner and outer parts of $Z(f; z)$. It can be proved that in the interior of the unit disc ($|z| < 1$) the absolute value of the inner part is bounded by 1:

$$|Z_{in}(f; z)| = |Z(f; z) \exp[-\Phi(f; z)]| \leqslant 1 \tag{37}$$

and that on the unit circle ($|z| = 1$) this absolute value is equal 1:

$$|\Theta_{in}(f; \theta)| = |\Theta(f; \theta) \exp[-\Phi(f; \theta)]| = |\Theta(f; \theta)| \exp[-\Phi_R(f; \theta)] = 1. \tag{38}$$

We use the notation $\Theta(f; \theta)$, $\Theta_{in}(f; \theta)$, $\Phi(f; \theta)$, etc, for the boundary functions of $Z(f; z)$, $Z_{in}(f; z)$, $\Phi(f; z)$, etc, obtained in the limit $|z| \to 1$ (with $z = |z|e^{i\theta}$). Relations (37) and (38) form a necessary and sufficient condition for a function $Z_{in}(f; z)$ to be an inner function. It follows from equation (38) that the phase distribution

$$P(f; \theta) = \frac{1}{2\pi}|\Theta(f; \theta)|^2 = \frac{1}{2\pi}|\Theta_{out}(f; \theta)|^2 = \frac{1}{2\pi}\exp[2\Phi_R(f; \theta)] \tag{39}$$

depends only on the outer part of the function and more specifically on $\Phi_R(f; \theta)$. Conversely, equation (33) shows that the $P(f; \theta)$ defines uniquely the outer part $Z_{out}(f; z)$.

The functions $\Phi_R(f; z)$, $\Phi_I(f; z)$ are real harmonic functions and they are harmonic conjugate of each other. The function $\ln|Z(f; z)|$ is a subharmonic function. Indeed, its boundary function is

$$\ln|\Theta(f; \theta)| = \ln|\exp[\Phi(f; \theta)]| = \Phi_R(f; \theta) = \tfrac{1}{2}\ln[2\pi P(f; \theta)] \tag{40}$$

while in the interior of the unit disc

$$\ln|Z(f; z)| = \ln|Z_{in}(f; z)| + \ln|Z_{out}(f; z)| \leqslant \ln|Z_{out}(f; z)| = \Phi_R(f; z). \tag{41}$$

A function which is identical with its inner (outer) part is called an inner (outer) function and the corresponding quantum state is called an inner (outer) state. It is interesting that all the properties of an outer state are determined by its phase distribution. Obviously, any two states with the same outer part have identical phase properties. For example, any inner state has the uniform phase distribution $(2\pi)^{-1}$. The number states $|n\rangle$ represented by the functions $z^n$ are inner states. An easy way to check whether a state is outer is to calculate the outer part $Z_{out}(f; z)$ from equations (32), (33) and to compare it with the whole function $Z(f; z)$. However, there is a simpler criterion for outer functions. It can be proved [1, 2] that a function $\exp[\Phi(z)]$ is an outer function if and only if

$$\Phi_R(z = 0) = \ln|\exp[\Phi(z = 0)]| = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \; \ln|\exp[\Phi(\theta)]|$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \; \Phi_R(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi} d\theta \; \ln[2\pi P(\theta)]. \tag{42}$$

By using this theorem, equation (12) and the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \ \ln\left|\mathrm{e}^{\mathrm{i}\theta} - \frac{1}{z_0}\right| = -\ln|z_0| \qquad |z_0| < 1 \tag{43}$$

we can prove that the SU(1, 1) coherent states $|z_0\rangle$ are outer states. One can also check, by using the generating function for the Gegenbauer polynomials [35], that the straightforward evaluation of the integral in (33) yields $Z(z_0; z) = Z_{\mathrm{out}}(z_0; z)$.

## 4. Factorization in the Barut–Girardello representation

The factorization into inner and outer parts has been introduced in the context of analytic representations in the unit disc. In this section we discuss how this factorization can be generalized for the Barut–Girardello representation [22]. Although the factorization is intimately connected with the analytic representations in the unit disc, it is nice to demonstrate that it can also be implemented in other representations. We choose the Barut–Girardello representation because there is a simple transformation that connects it with the analytic representation in the unit disc. The Barut–Girardello states are defined as the eigenstates of the SU(1, 1) lowering generator $\hat{K}_-$ [22]:

$$\hat{K}_-|u\rangle = u|u\rangle. \tag{44}$$

Here $u$ is an arbitrary complex number. Various properties of these states were studied in different physical contexts [15, 36, 37, 38, 39]. In the Holstein–Primakoff realization (1) the Barut–Girardello states are given by

$$|u\rangle = [I_0(2|u|)]^{-1/2}\sum_{n=0}^{\infty}\frac{u^n}{n!}|n\rangle \tag{45}$$

where $I_0(x)$ is the zero-order Bessel function of the first kind. The analytic representation of these states in the unit disc is

$$Z(u; z) = (1-|z|^2)^{-1/2}\langle u|z\rangle = [I_0(2|u|)]^{-1/2}\exp(u^* z). \tag{46}$$

By using relation (42) or by the straightforward evaluation of the integral in (33), we can prove that the Barut–Girardello states $|u\rangle$ are outer states. The states $|u\rangle$ form an overcomplete set in the Fock space with the resolution of the identity

$$\int \mathrm{d}\mu(u)\ |u\rangle\langle u| = \hat{1} \qquad \mathrm{d}\mu(u) = \frac{2}{\pi}K_0(2|u|)I_0(2|u|)\mathrm{d}^2u. \tag{47}$$

The integration in (47) is over the whole $u$ plane, and $K_0(x)$ is the zero-order Bessel function of the second kind. One can define the Barut–Girardello analytic representation in the complex plane [22] (which is different from the familiar Bargmann representation [40]). A normalized state $|f\rangle$ is represented by the function

$$U(f; u) = [I_0(2|u|)]^{1/2}\langle f|u\rangle = \sum_{n=0}^{\infty} f_n^*\frac{u^n}{n!} \tag{48}$$

$$|f\rangle = \int \mathrm{d}\mu(u)\ [I_0(2|u|)]^{-1/2}U^*(f; u)|u\rangle. \tag{49}$$

It can be shown [26] that the Barut–Girardello representation and the representation in the unit disc are related through the '$\mathcal{L}$ transformation' (which is effectively the Laplace transform):

$$Z(f; z) = \frac{1}{z}\mathcal{L}[U(f; u)] = \frac{1}{z}\int_0^{\infty}\mathrm{d}u\ U(f; u)\exp(-u/z) \tag{50}$$

where the integration is along the positive real semiaxis and $z$ belongs to the right half of the unit disc ($\mathrm{Re}\, z > 0$). This equation defines $Z(f; z)$ in the right half of the unit disc and through analytic continuation in the whole unit disc. The inverse transformation is

$$U(f; u) = \mathcal{L}^{-1}[zZ(f; z)] = \frac{1}{2\pi \mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \mathrm{d}w \, \frac{Z(f; 1/w)}{w} \exp(wu). \tag{51}$$

The integration is along the line $1 + \mathrm{i}t$ where $t$ is a real number. If we define the functions

$$U_{\mathrm{in}}(f; u) = \mathcal{L}^{-1}[zZ_{\mathrm{in}}(f; z)] \tag{52}$$

$$U_{\mathrm{out}}(f; u) = \mathcal{L}^{-1}[Z_{\mathrm{out}}(f; z)] \tag{53}$$

the substitution of equation (31) in (51) defines the factorization in the context of the Barut–Girardello representation. Note that the factorization here is a convolution:

$$U(f; u) = \int_0^u \mathrm{d}x \; U_{\mathrm{in}}(f; x) U_{\mathrm{out}}(f; u - x). \tag{54}$$

The integration is along the line from 0 to the complex number $u$. As an example we consider the number states $|n\rangle$ which are inner states with $Z_{\mathrm{out}}(n; z) = 1$, $Z_{\mathrm{in}}(n; z) = z^n$. Then equations (51)–(53) give

$$U_{\mathrm{out}}(n; u) = 2\delta(u) \qquad U(n; u) = U_{\mathrm{in}}(n; u) = \frac{u^n}{n!}. \tag{55}$$

(We use the convention in which the delta function $\delta(u)$ is symmetrical about $u = 0$.) For an outer state $|f\rangle$ with $Z(f; z) = Z_{\mathrm{out}}(f; z)$, we obtain

$$U_{\mathrm{in}}(f; u) = 1 \qquad U(f; u) = \int_0^u \mathrm{d}x \; U_{\mathrm{out}}(f; x). \tag{56}$$

Then the outer part of the Barut–Girardello representation can be written as

$$U_{\mathrm{out}}(f; u) = 2 f_0^* \delta(u) + \partial U(f; u)/\partial u. \tag{57}$$

## 5. Examples: quantum superposition states

Quantum superposition states have many interesting properties such as squeezing and sub-Poissonian photon statistics [41]. These features arise because components of a superposition state interfere with each other in phase space. Here we show how this interference in phase space can be examined via the factorization of the analytic representation in the unit disc. In particular we show how states whose analytic representation in the unit disc is a 'Blaschke factor' can be produced as quantum superpositions.

We start by showing that a superposition of two inner states can be an outer state. It can be proved (see exercise 12 in chapter 5 of [2]) that if $Z(f; z)$ is an inner function then $Z(f; z) + 1$ is an outer function. A superposition of an inner state $|f\rangle$ and the vacuum of the form $A(|f\rangle + |0\rangle)$ (where $A$ is a normalization constant) is represented by the analytic function $A[Z(f; z) + 1]$ that is, by virtue of the above statement, an outer function (the multiplication by a constant leaves it outer). Therefore, we can formulate the following theorem: *the superposition of any inner state and the vacuum is an outer state*. For example, the superposition of the vacuum and any other number state, $|n\rangle_{\mathrm{out}} = (|0\rangle + |n\rangle)/\sqrt{2}$, is an outer state because its analytic function $(1 + z^n)/\sqrt{2}$ is an outer function. All physical properties of such a state are determined by its phase distribution $(1 + \cos n\theta)/2\pi$.

We also show how an inner state can be produced as a quantum superposition. We consider the superposition of the SU(1, 1) coherent state $|z_0\rangle$ (that is an outer state) and its first shifted state $|z_0\rangle_1 = \hat{E}_+|z_0\rangle$ (these states will be studied in detail in section 6.1). This superposition is defined as

$$|z_0\rangle_{\rm in} = (1-|z_0|^2)^{-1/2}(|z_0\rangle_1 - z_0^*|z_0\rangle) = (-z_0^*)|0\rangle + (1-|z_0|^2)\sum_{n=1}^{\infty} z_0^{n-1}|n\rangle. \tag{58}$$

The corresponding analytic function

$$Z^{(\rm in)}(z_0; z) = \frac{z-z_0}{1-z_0^* z} \tag{59}$$

is a Blaschke factor [1, 2] and we refer to the state (58) as a Blaschke state. It can be easily proved that the function (59) is an inner function. Therefore the Blaschke state (58) is an inner state, though it is composed of an outer state and a shifted outer state. In this case interference in phase space produces a state with a uniform phase distribution.

An interesting example is the superposition of the SU(1, 1) coherent states $|z_0\rangle$ and $|-z_0\rangle$, defined as

$$|z_0,\tau\rangle = \mathcal{N}^{-1/2}(|z_0\rangle + {\rm e}^{{\rm i}\tau}|-z_0\rangle) \tag{60}$$

where $\mathcal{N}$ is the normalization factor

$$\mathcal{N} = 2\left[1 + \frac{1-|z_0|^2}{1+|z_0|^2}\cos\tau\right] \tag{61}$$

and $-\pi \leqslant \tau \leqslant \pi$. The analytic function in the unit disc corresponding to the superposition state $|z_0,\tau\rangle$ is

$$Z(z_0,\tau; z) = \mathcal{A}\frac{\cos(\tau/2) + {\rm i}\sin(\tau/2)z_0^* z}{1-z_0^{*2}z^2} \tag{62}$$

$$\mathcal{A} \equiv 2\mathcal{N}^{-1/2}(1-|z_0|^2)^{1/2}\exp(-{\rm i}\tau/2). \tag{63}$$

Important information about an analytic function can be obtained by investigating its zeros. It is clear that any outer function has no zeros in the unit disc. An inner function can be written as a product $S(z)B(z)$ where $S(z)$ is a function with no zeros (known as singular function) and $B(z)$ is a function with zeros. The function $B(z)$ with zeros $\{\gamma_k\}$ can be written as a Blaschke product [1, 2]:

$$B(z) = \prod_k \left(\frac{\gamma_k^*}{|\gamma_k|}\frac{\gamma_k - z}{1-\gamma_k^* z}\right)^{p_k} \tag{64}$$

where $p_k$ are non-negative integers and $\gamma_k$ are distinct numbers in the unit disc. We see that the function $Z(z_0,\tau;z)$ of equation (62) has only one zero

$$\gamma = \frac{{\rm i}\cot(\tau/2)}{z_0^*} \tag{65}$$

that lies in the unit disc for $|z_0| > |\cot(\tau/2)|$. We next calculate the outer part of the function $Z(z_0,\tau;z)$ using equation (33). In the case $|z_0| < |\cot(\tau/2)|$ (for example, for $|\tau| \leqslant \pi/2$), we find $Z_{\rm out}(z_0,\tau;z) = Z(z_0,\tau;z)$; so the superposition state $|z_0,\tau\rangle$ with $|z_0| < |\cot(\tau/2)|$ is an outer state. This is consistent with the fact that in this case the function $Z(z_0,\tau;z)$ has no zeros in the unit disc. When $|z_0| > |\cot(\tau/2)|$ (that is possible only for $|\tau| > \pi/2$), equation (33) gives

$$Z_{\rm out}(z_0,\tau;z) = \mathcal{A}\frac{z_0^*}{|z_0|}\frac{z_0\sin(\tau/2) + {\rm i}z\cos(\tau/2)}{1-z_0^{*2}z^2}. \tag{66}$$

Then the inner part is

$$Z_{\rm in}(z_0, \tau; z) = \frac{Z(z_0, \tau; z)}{Z_{\rm out}(z_0, \tau; z)} = \frac{\gamma^*}{|\gamma|} \frac{\gamma - z}{1 - \gamma^* z} \tag{67}$$

with $\gamma$ given by equation (65). We see that in this case the inner part is exactly a Blaschke factor that has a zero in the unit disc.

## 6. Phase space for number and phase variables

The purpose of this section is to demonstrate that the factorization into inner and outer parts is intimately connected with the phase-space formalism based on the number and phase variables. In section 6.1 we study shifted states generated by the exponential phase operator $\hat{E}_+$ and in section 6.2 we consider the number-phase Wigner function.

### *6.1. Shifted states*

We consider a state $|f\rangle$ with the analytic function $Z(f; z)$ and define the state $|g\rangle$ as

$$|g\rangle = \hat{W}(m, \beta, \gamma)|f\rangle. \tag{68}$$

It is easily seen that the analytic function $Z(g; z)$ of the state $|g\rangle$ is given by

$$Z(g; z) = \mathrm{e}^{-\mathrm{i}\gamma} z^m Z(f; z\mathrm{e}^{-\mathrm{i}\beta}) \tag{69}$$

$$Z_{\rm in}(g; z) = \mathrm{e}^{-\mathrm{i}\gamma} z^m Z_{\rm in}(f; z\mathrm{e}^{-\mathrm{i}\beta}) \tag{70}$$

$$\Phi(g; z) = \Phi(f; z\mathrm{e}^{-\mathrm{i}\beta}). \tag{71}$$

The corresponding boundary functions are related as

$$\Theta(g; \theta) = \mathrm{e}^{\mathrm{i}m\theta - \mathrm{i}\gamma} \Theta(f; \theta - \beta) \tag{72}$$

$$\Theta_{\rm in}(g; \theta) = \mathrm{e}^{\mathrm{i}m\theta - \mathrm{i}\gamma} \Theta_{\rm in}(f; \theta - \beta) \tag{73}$$

$$\Phi(g; \theta) = \Phi(f; \theta - \beta). \tag{74}$$

When $\beta = 0$, the operator $\hat{W}$ changes only the inner part of a state. Therefore, the phase distribution of a state is invariant under the action of the operator $\hat{E}_+^m$. For an outer state $|f\rangle$ with $Z(f; z) = Z_{\rm out}(f; z)$ we can define the sequence $\{|f\rangle_m\}$ of the shifted states

$$|f\rangle_m = \hat{E}_+^m |f\rangle = \sum_{n=m}^{\infty} f_{n-m} |n\rangle \qquad m = 0, 1, 2, \ldots. \tag{75}$$

All these states have identical outer functions $Z(f; z)$ and phase distributions $P(f; \theta)$. The inner function of the shifted state $|f\rangle_m$ is simply $z^m$. Therefore the number distribution is shifted as

$$P^{(m)}(f; n) = |\langle n|f\rangle_m|^2 = \begin{cases} P(f; n-m) & n \geqslant m \\ 0 & n < m. \end{cases} \tag{76}$$

The Barut–Girardello representation of the shifted state $|f\rangle_m$ is given by

$$U_{\rm in}^{(m)}(f; u) = \frac{u^m}{m!} \qquad U^{(m)}(f; u) = \frac{1}{m!} \int_0^u \mathrm{d}x\, (u - x)^m U_{\rm out}(f; x). \tag{77}$$

Using equation (57), we can express $U^{(m)}(f; u)$ with $m \geqslant 1$ as an integral over $U(f; u)$:

$$U^{(m)}(f; u) = \frac{1}{(m-1)!} \int_0^u \mathrm{d}x\, (u - x)^{m-1} U(f; x). \tag{78}$$

For example, the SU(1, 1) coherent state $|z\rangle$ generates the sequence $\{|z\rangle_m\}$ of the shifted states

$$|z\rangle_m = \hat{E}_+^m |z\rangle = \sqrt{1-|z|^2} \sum_{n=m}^{\infty} z^{n-m} |n\rangle \tag{79}$$

which satisfy the eigenvalue equation

$$\left(\hat{E}_- - |m-1\rangle\langle m|\right) |z\rangle_m = z|z\rangle_m. \tag{80}$$

The Barut–Girardello state $|u\rangle$ generates the sequence $\{|u\rangle_m\}$ of the shifted states

$$|u\rangle_m = \hat{E}_+^m |u\rangle = [I_0(2|u|)]^{-1/2} \sum_{n=m}^{\infty} \frac{u^{n-m}}{(n-m)!} |n\rangle \tag{81}$$

which satisfy the eigenvalue equation

$$\hat{E}_- \left(\hat{N} - m\right) |u\rangle_m = u|u\rangle_m. \tag{82}$$

The states $|u\rangle_m$ have been recently considered as the generalized philophase states [15].

### *6.2. The number-phase Wigner function*

For a quantum state $|f\rangle$ with the number representation $f_n$ and the phase representation $\Theta(f;\theta)$, the number-phase Wigner function is defined as [29]

$$\begin{aligned} S(f;n,\theta) &= \frac{1}{2\pi}\left[\sum_{p=-n}^{n} e^{i2p\theta} f_{n-p} f_{n+p}^* + \sum_{p=-n}^{n-1} e^{i(2p+1)\theta} f_{n-p-1} f_{n+p}^*\right] \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d\phi\ (1+e^{i\phi}) e^{-i2n\phi} \Theta^*(f;\theta-\phi)\Theta(f;\theta+\phi). \end{aligned} \tag{83}$$

The number and phase distributions are related to the number-phase Wigner function through the relations

$$P(f;n) = |\langle f|n\rangle|^2 = \int_{-\pi}^{\pi} d\theta\ S(f;n,\theta) \tag{84}$$

$$P(f;\theta) = \frac{1}{2\pi} |\langle f|\theta\rangle|^2 = \sum_{n=0}^{\infty} S(f;n,\theta). \tag{85}$$

The number-phase Wigner function of an outer state is uniquely determined by its phase distribution $P(f;\theta)$. The number-phase Wigner function for the state $|g\rangle$ of equation (68) is given by

$$S(g;n,\theta) = \begin{cases} S(f;n-m,\theta-\beta) & n \geqslant m \\ 0 & n < m. \end{cases} \tag{86}$$

We calculate the number-phase Wigner function for some states discussed above. The number (Fock) state $|m\rangle$ (that is an inner state) is represented by the function

$$S(m;n,\theta) = \frac{1}{2\pi} \delta_{n,m}. \tag{87}$$

For the superposition state $|m\rangle_{\mathrm{out}} = (|0\rangle + |m\rangle)/\sqrt{2}$ (that is an outer state), one obtains [29]

$$S^{(\mathrm{out})}(m;n,\theta) = \frac{1}{4\pi} [\delta_{n,0} + \delta_{n,m} + 2\delta_{n,k} \cos(m\theta)] \tag{88}$$

where $k = m/2$ if $m$ is even and $k = (m+1)/2$ if $m$ is odd. The SU(1, 1) coherent state $|z\rangle$ with $z = |z|e^{i\phi}$ is represented by the function

$$S(z; n, \theta) = \frac{1-|z|^2}{2\pi} \left\{ |z|^{2n} U_{2n}[\cos(\theta-\phi)] + |z|^{2n-1} U_{2n-1}[\cos(\theta-\phi)] \right\} \tag{89}$$

where $U_n(x)$ is the $n$-order Chebyshev polynomial of the second kind:

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{90}$$

and $U_n(x) \equiv 0$ for $n < 0$. For the Barut–Girardello state $|u\rangle$ with $u = |u|e^{i\varphi}$, we find

$$S(u; n, \theta) = \frac{1}{2\pi I_0(2|u|)} \left\{ \frac{[2|u|\cos(\theta-\varphi)]^{2n}}{(2n)!} + \frac{[2|u|\cos(\theta-\varphi)]^{2n-1}}{(2n-1)!} \right\}. \tag{91}$$

The second term in equations (89), (91) should be omitted for $n = 0$. The number-phase Wigner functions for the shifted states $|z\rangle_m$ and $|u\rangle_m$ are obtained from equations (89) and (91), respectively, according to relation (86). For the superposition state $|z\rangle_{\mathrm{in}}$ of equation (58) (that is an inner state) the number-phase Wigner function can be written in terms of the Chebyshev polynomials:

$$\begin{aligned} S^{(\mathrm{in})}(z; n, \theta) = \frac{1}{2\pi} \left\{ r^{2n-3} U_{2n-3}[\cos(\theta-\phi)] + [1 - 2r\cos(\theta-\phi)] r^{2n-2} U_{2n-2}[\cos(\theta-\phi)] \right. \\ \left. + [r^2 - 2r\cos(\theta-\phi)] r^{2n-1} U_{2n-1}[\cos(\theta-\phi)] + r^{2n+2} U_{2n}[\cos(\theta-\phi)] \right\} \end{aligned} \tag{92}$$

where $z = re^{i\phi}$. As we have shown, the superposition state $|z, \tau\rangle$ of equation (60) is an outer state for $|z| < |\cot(\tau/2)|$, while for $|z| > |\cot(\tau/2)|$ it has an inner part that is a Blaschke factor (see equation (67)). The state $|z, \tau\rangle$ with $z = re^{i\phi}$ is represented by the number-phase Wigner function

$$\begin{aligned} S(z, \tau; n, \theta) = \frac{|\mathcal{A}|^2}{2\pi} \left\{ [\cos^2(\tau/2)] r^{2n} U_n[\cos 2(\theta-\phi)] \right. \\ \left. + [r^2 \sin^2(\tau/2) - r\sin(\tau)\sin(\theta-\phi)] r^{2(n-1)} U_{n-1}[\cos 2(\theta-\phi)] \right\}. \end{aligned} \tag{93}$$

For $r < |\cot(\tau/2)|$ this function is uniquely determined by its marginal distribution $P(z, \tau; \theta)$; while for $r > |\cot(\tau/2)|$ there exists the inner part that brings new features which cannot be described by the phase distribution alone.

## 7. Conclusions

We have shown that the factorization of analytic representations in the unit disc into inner and outer parts has an important physical significance. It is intimately related to the number-phase Weyl semigroup and the number-phase quantum statistical properties. This factorization can be implemented not only in analytic representations in the unit disc, but also in other representations.

We have given several examples of inner and outer states and we have constructed explicitly Blaschke states. Inner states have a uniform phase distribution while outer states are uniquely determined by their phase properties. This work has demonstrated that pure mathematical concepts in the theory of Hardy spaces have an interesting physical interpretation.

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